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Integrating Cognitive Decision-Making and Production Workflows with HyperFuzzy Trace and HyperFuzzy Front-End

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Abstract

Fuzzy sets model vagueness by assigning each element a membership degree in the interval $[0, 1]$. Hyperfuzzy sets generalize this notion by mapping each element to a nonempty subset of $[0, 1]$, thereby accommodating both uncertainty and variability in membership degrees. An (m, n)-superhyperfuzzy set further extends these frameworks by assigning to each nonempty member of the m-th and n-th iterated powersets a nonempty family of subsets of $[0, 1]$, enabling hierarchical and nested imprecision. These generalized fuzzy models have found applications in decision-making and many other domains. Meanwhile, the theories of Fuzzy Trace—retaining both verbatim and gist representations of information—and Fuzzy Front-Ends—using graded membership functions to evaluate and aggregate multiple criteria for soft go/no-go decisions—play a crucial role in information processing and innovation screening. However, corresponding concepts built upon hyperfuzzy and superhyperfuzzy structures have not yet been explored. To bridge this gap, we introduce extensions of Fuzzy Trace and Fuzzy Front-Ends based on hyperfuzzy and (m, n)-superhyperfuzzy sets. We present their formal definitions, examine basic properties, and illustrate them with examples. By doing so, we open the door to applying Fuzzy Trace and Fuzzy Front-End methodologies in contexts characterized by hierarchical uncertainty.

Keywords: Fuzzy set, Fuzzy trace, Hyperfuzzy trace, Hyperfuzzy front-end, Fuzzy front-end.

1 | Preliminaries

Throughout this paper, all sets are assumed finite. We introduce the basic notation and concepts used in what follows.

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1.1 | HyperFuzzy Sets and SuperHyperFuzzy Sets

Fuzzy sets model imprecision by assigning each element a membership value in $[0, 1]$ [1]. Numerous extensions have appeared—Bipolar Fuzzy Sets [2], Intuitionistic Fuzzy Sets [3], Neutrosophic Sets [4], Hesitant Fuzzy Sets [5], QuadriPartitioned Neutrosophic Sets[6, 7], Spherical Fuzzy Sets[8], and Plithogenic Sets [9, 10]. In this work, we concentrate on two higher-order generalizations: the *HyperFuzzy Set* [11, 12], which maps each element to a nonempty subset of $[0, 1]$, and its hierarchical extension, the *SuperHyperFuzzy Set* [13, 14], which employs iterated powerset constructions to capture multiple layers of uncertainty.

Definition 1 (Base Set)**.** A *base set* S is the primary universe under consideration:

$$S = \{\, x \mid x \text{ belongs to the domain of discourse}\}.$$

All subsequent constructions—powersets, iterated powersets, and their nonempty variants—are built from S.

Definition 2 (Powerset)**.** [15] The *powerset* of S, denoted $\mathcal{P}(S)$, is the collection of all subsets of S, including the empty set:

$$\mathcal{P}(S) = \{\, A \mid A \subseteq S\}.$$

Definition 3 (n-th Powerset)**.** [16] Define $\mathcal{P}_1(H) = \mathcal{P}(H)$. For $n \geq 1$, the $(n+1)$*-th powerset* is

$$\mathcal{P}_{n+1}(H) \;=\; \mathcal{P}\big(\mathcal{P}_n(H)\big).$$

The *nonempty* n*-th powerset* is obtained by removing the empty set at each stage:

$$\mathcal{P}_1^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}, \quad \mathcal{P}_{n+1}^*(H) = \mathcal{P}\big(\mathcal{P}_n^*(H)\big) \setminus \{\emptyset\}.$$

Definition 4 (Fuzzy Set)**.** [1] A *fuzzy set* τ on a nonempty universe Y is a function

$$\tau : Y \;\longrightarrow\; [0, 1].$$

A *fuzzy relation* δ on Y is a fuzzy subset of $Y \times Y$ satisfying

$$\delta(y, z) \;\leq\; \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 5 (HyperFuzzy Set)**.** [11, 17] Let X be a nonempty universe. A *HyperFuzzy Set* $\tilde{A}$ on X is given by

$$\tilde{\mu} : X \;\longrightarrow\; \tilde{\mathcal{P}}([0, 1]),$$

where $\tilde{\mathcal{P}}([0, 1])$ denotes all nonempty subsets of $[0, 1]$. For each $x \in X$, the set $\tilde{\mu}(x) \subseteq [0, 1]$ represents the range of plausible membership degrees for x, thus modeling both fuzziness and variability simultaneously.

Example 1 (HyperFuzzy Set: Smartphone Satisfaction Variability)**.** Let

$$X = \{\text{iPhone } 14,\ \text{Galaxy } S23,\ \text{Pixel } 7\}$$

be a set of popular smartphone models. We conduct two independent user surveys—one by tech reviewers and one by general consumers—each reporting satisfaction scores in $[0, 1]$. Define the hyper-membership function

$$\tilde{\mu} : X \;\longrightarrow\; \tilde{\mathcal{P}}([0, 1]),$$

where

$$\tilde{\mu}(\text{iPhone } 14) = \{\, 0.85, 0.90\}, \quad \tilde{\mu}(\text{Galaxy } S23) = \{\, 0.80, 0.88\}, \quad \tilde{\mu}(\text{Pixel } 7) = \{\, 0.75, 0.82\}.$$

Here for each model x, the set $\tilde{\mu}(x) \subseteq [0, 1]$ collects both the expert-review average and the consumer-survey average. This HyperFuzzy Set captures not only the degree of satisfaction ("fuzziness") but also the variability between different user groups.

Definition 6 (m, n-SuperHyperFuzzy Set)**.** [18] Let X be a nonempty set and let $m, n \in \mathbb{N}_0$. Define the nonempty k-th powerset of a set Y by

$$\mathcal{P}_0^*(Y) = Y, \quad \mathcal{P}_k^*(Y) = \mathcal{P}\big(\mathcal{P}_{k-1}^*(Y)\big) \setminus \{\emptyset\}, \quad k \geq 1.$$

In particular, $\mathcal{P}_m^*(X)$ is the family of all nonempty elements of the m-th iterated powerset of X, and $\mathcal{P}_n^*([0,1])$ is defined analogously. Then an (m, n)-SuperHyperFuzzy Set on X is a function

$$\tilde{\mu}_{m,n} : \mathcal{P}_m^*(X) \longrightarrow \tilde{\mathcal{P}}_n^*\big([0,1]\big), \quad A \mapsto \tilde{\mu}_{m,n}(A),$$

where $\tilde{\mathcal{P}}_n^*([0,1])$ denotes the collection of all nonempty subsets of $\mathcal{P}_n([0,1])$. Thus each $A \in \mathcal{P}_m^*(X)$ is assigned a nonempty family of membership-degree sets $\tilde{\mu}_{m,n}(A) \subseteq \mathcal{P}_n([0,1])$, capturing hierarchical uncertainty across both the m- and n-levels.

Example 2 (Real-World $(1, 2)$-SuperHyperFuzzy Set: Sustainable Practice Bundles)**.** In a municipal sustainability assessment, let

$$X = \{\text{Recycling},\ \text{SolarPanels},\ \text{WaterHarvest}\}$$

be three core green practices. Choose $m = 1$, $n = 2$. Then

$$\mathcal{P}_1^*(X) = \mathcal{P}(X) \setminus \{\emptyset\} \ = \ \big\{\{\text{Recycling}\}, \{\text{SolarPanels}\}, \{\text{WaterHarvest}\}, \{\text{Recycling}, \text{SolarPanels}\}, \dots\big\},$$

and

$$\mathcal{P}_2^*([0,1]) = \mathcal{P}\big(\mathcal{P}([0,1])\big) \setminus \{\emptyset\},$$

the collection of all nonempty families of numeric membership-sets in $[0, 1]$.

Define

$$\tilde{\mu}_{1,2} : \mathcal{P}_1^*(X) \longrightarrow \mathcal{P}_2^*([0,1])$$

by the following representative assignments:

Practice bundle $\{\text{Recycling}, \text{SolarPanels}\}$**:**

$$\tilde{\mu}_{1,2}\big(\{\text{Recycling}, \text{SolarPanels}\}\big) = \big\{\,[0.7, 0.9],\ [0.6, 0.8]\big\} \subseteq \mathcal{P}([0,1]),$$

where

- $[0.7, 0.9]$ represents expert-elicited membership degrees for combined recycling+solar adoption;
- $[0.6, 0.8]$ reflects community survey–based degrees.

Practice bundle $\{\text{WaterHarvest}\}$**:**

$$\tilde{\mu}_{1,2}\big(\{\text{WaterHarvest}\}\big) = \big\{\,[0.5, 0.7],\ [0.4, 0.6]\big\},$$

where

- $[0.5, 0.7]$ is the range from field-monitoring data;
- $[0.4, 0.6]$ captures expert uncertainty in dry-season performance.

One checks immediately that $\tilde{\mu}_{1,2}$ indeed maps each nonempty subset of X to a nonempty family of subsets of $[0, 1]$, so it is a valid $(1, 2)$-SuperHyperFuzzy Set on X. This concrete example encodes two hierarchical layers of uncertainty—expert vs. survey vs. field data—over each bundle of sustainable practices.

2 |Result: Fuzzy Trace

2.1 |Fuzzy Trace

A fuzzy trace retains both verbatim and gist memory of information items, encoding literal content and bottom-line meaning, respectively (cf.[19, 20]).

Definition 7. (Fuzzy Trace)**.** (cf.[21, 22, 23]) Let $\mathcal{I}$ be a collection of information stimuli (e.g. sentences, numbers, images). A *fuzzy trace* of an item $x \in \mathcal{I}$ is the ordered pair

$$\mathrm{FT}(x) \;=\; \big(v(x),\, g(x)\big),$$

where

- $v\colon \mathcal{I} \to \mathcal{V}$ is the *verbatim encoding*, preserving the exact, literal form of x (e.g. precise numbers, surface wording, graphical details);
- $g\colon \mathcal{I} \to \mathcal{G}$ is the *gist encoding*, capturing only the bottom–line meaning of x in a possibly coarse, qualitative form.

We denote the stored memory as

$$\mathcal{M} \;=\; \big\{\mathrm{FT}(x) \mid x \in \mathcal{I}\big\}.$$

Remark 1. In many applications, the gist space $\mathcal{G}$ admits a hierarchy of granularity levels

$$\mathcal{G}_0 \;\subset\; \mathcal{G}_1 \;\subset\; \cdots \subset\; \mathcal{G}_k,$$

where $\mathcal{G}_0$ is purely categorical (e.g. “some vs. none”), $\mathcal{G}_1$ is ordinal (e.g. “low vs. medium vs. high”), and higher $\mathcal{G}_i$ allow finer distinctions (interval or ratio-level interpretations). The function

$$g(x) \;=\; \big(g_0(x),\, g_1(x),\, \ldots,\, g_k(x)\big)$$

thus encodes multiple gist representations in parallel.

Example 3. (Fuzzy Trace: Clinical Test Recall)**.** (cf.[24]) Let

$$\mathcal{I} = \big\{\text{"Blood pressure: 145/90 mmHg"},\ \text{"Fasting glucose: 110 mg/dL"}$$

be a patient’s clinical test reports. Define the verbatim space

$$\mathcal{V} = \{\text{all exact numeric readings with units}\},$$

and the gist space

$$\mathcal{G} = \{\text{Normal, Elevated, High}\},$$

using standard medical thresholds.

For each $x \in \mathcal{I}$, set:

$$v\big(\text{"Blood pressure: 145/90 mmHg"}\big) = 145/90 \text{ mmHg},$$

$$g\big(\text{"Blood pressure: 145/90 mmHg"}\big) = \text{High (Stage I Hypertension)},$$

$$v\big(\text{"Fasting glucose: 110 mg/dL"}\big) = 110 \text{ mg/dL},$$

$$g\big(\text{"Fasting glucose: 110 mg/dL"}\big) = \text{Elevated (Prediabetes)}.$$

Hence the fuzzy traces are

$$\mathrm{FT}\big(\text{"Blood pressure: 145/90 mmHg"}\big) = \big(145/90 \text{ mmHg, High}\big),$$

$$\mathrm{FT}\big(\text{"Fasting glucose: 110 mg/dL"}\big) = \big(110 \text{ mg/dL, Elevated}\big).$$

The stored memory is

$$\mathcal{M} = \big\{\big(145/90 \text{ mmHg, High}\big),\ \big(110 \text{ mg/dL, Elevated}\big)\big\}.$$

2.2 | HyperFuzzy Trace

A hyperfuzzy trace generalizes fuzzy trace by storing sets of possible verbatim and gist encodings for each item, capturing uncertainty.

Definition 8 (HyperFuzzy Trace)**.** Let $\mathcal{I}$ be a set of information items, let $\mathcal{V}$ be the verbatim-encoding space, and let $\mathcal{G} \subseteq [0,1]$ be the gist-encoding space. Denote by

$$\tilde{\mathcal{P}}(X) = \{\, A \subseteq X \mid A \neq \varnothing\}$$

the collection of all nonempty subsets of a set X. A *HyperFuzzy Trace* is a pair of functions

$$\tilde{v} : \mathcal{I} \longrightarrow \tilde{\mathcal{P}}(\mathcal{V}), \qquad \tilde{g} : \mathcal{I} \longrightarrow \tilde{\mathcal{P}}(\mathcal{G}),$$

assigning to each $x \in \mathcal{I}$ a nonempty *verbatim hyper-set* $\tilde{v}(x) \subseteq \mathcal{V}$ and a nonempty *gist hyper-set* $\tilde{g}(x) \subseteq \mathcal{G}$. We write

$$\mathrm{HFT}(x) \;=\; \big(\tilde{v}(x),\, \tilde{g}(x)\big),$$

and call the family

$$\mathcal{H} = \big\{\mathrm{HFT}(x) \mid x \in \mathcal{I}\big\}$$

Example 4 (HyperFuzzy Trace: Clinical Test Variability)**.** Let

$$\mathcal{I} = \big\{\text{"Blood pressure reading"},\ \text{"Fasting glucose reading"}\big\}.$$

Define the *verbatim-encoding space*

$$\mathcal{V} = \{\text{all plausible numeric readings with units}\}$$

and the *gist-encoding space*

$$\mathcal{G} \subseteq [0,1]$$

where each $g \in \mathcal{G}$ represents a degree of clinical risk.

We model measurement uncertainty by assigning each test two possible readings and two possible risk-level degrees:

Blood pressure:

$$\tilde{v}\big(\text{"Blood pressure reading"}\big) = \big\{\text{"145/90 mmHg"},\ \text{"150/95 mmHg"}\big\},$$
$$\tilde{g}\big(\text{"Blood pressure reading"}\big) = \{0.7,\ 0.8\}.$$

Fasting glucose:

$$\tilde{v}\big(\text{"Fasting glucose reading"}\big) = \big\{\text{"110 mg/dL"},\ \text{"115 mg/dL"}\big\}, \quad \tilde{g}\big(\text{"Fasting glucose reading"}\big) = \{0.6,\ 0.7\}.$$

Thus the HyperFuzzy Traces are

$$\mathrm{HFT}\big(\text{"Blood pressure reading"}\big) = \Big(\{\text{"145/90 mmHg"}, \text{"150/95 mmHg"}\},\, \{0.7, 0.8\}\Big),$$

$$\mathrm{HFT}\big(\text{"Fasting glucose reading"}\big) = \Big(\{\text{"110 mg/dL"}, \text{"115 mg/dL"}\},\, \{0.6, 0.7\}\Big).$$

The stored family of HyperFuzzy Traces is

$$\mathcal{H} = \big\{\mathrm{HFT}(x) \mid x \in \mathcal{I}\big\}.$$

Theorem 1 *Let $(\tilde{v}, \tilde{g})$ be a HyperFuzzy Trace on $(\mathcal{I}, \mathcal{V}, \mathcal{G})$. Then:*

(1) ***Generalization of Fuzzy Trace.*** *If for all $x \in \mathcal{I}$ both $\tilde{v}(x)$ and $\tilde{g}(x)$ are singletons,*

$$\tilde{v}(x) = \{v_0(x)\}, \quad \tilde{g}(x) = \{g_0(x)\},$$

then the map $x \mapsto \big(v_0(x), g_0(x)\big)$ is exactly a classical fuzzy trace.

(2) ***HyperFuzzy Set Structure.*** *Each component*

$$\tilde{v} : \mathcal{I} \longrightarrow \tilde{\mathcal{P}}(\mathcal{V}), \qquad \tilde{g} : \mathcal{I} \longrightarrow \tilde{\mathcal{P}}(\mathcal{G})$$

is a HyperFuzzy Set on the universe $\mathcal{I}$, since it assigns to every $x \in \mathcal{I}$ a nonempty subset of $\mathcal{V}$ (resp. $\mathcal{G}$).

Proof: (1) If $\tilde{v}(x) = \{v_0(x)\}$ and $\tilde{g}(x) = \{g_0(x)\}$ for each x, then define

$$v(x) = v_0(x), \quad g(x) = g_0(x).$$

By construction, (v, g) is a fuzzy trace in the sense that $v : \mathcal{I} \to \mathcal{V}$ and $g : \mathcal{I} \to \mathcal{G}$ are single-valued encodings. Hence every HyperFuzzy Trace with singleton images reduces to a classical fuzzy trace.

(2) By definition of $\tilde{\mathcal{P}}(\mathcal{V})$, each $\tilde{v}(x) \subseteq \mathcal{V}$ is nonempty. Thus

$$\tilde{v} : \mathcal{I} \longrightarrow \tilde{\mathcal{P}}(\mathcal{V})$$

satisfies exactly the requirements of a HyperFuzzy Set on universe $\mathcal{I}$ (cf. Definition of HyperFuzzy Set). The same argument applies to $\tilde{g} : \mathcal{I} \to \tilde{\mathcal{P}}(\mathcal{G})$. Therefore both components carry the full HyperFuzzy Set structure.

□

Theorem 2. (Pushforward under Verbatim-Encoding Maps)**.** *Let $f : \mathcal{V} \to \mathcal{W}$ be any function to a set $\mathcal{W}$. Define*

$$f_\sharp(\tilde{v}(x)) = \{\, f(v) \mid v \in \tilde{v}(x)\} \subseteq \mathcal{W},$$

and set

$$\tilde{v}' : \mathcal{I} \longrightarrow \tilde{\mathcal{P}}(\mathcal{W}), \quad \tilde{v}'(x) = f_\sharp(\tilde{v}(x)).$$

Then $(\tilde{v}', \tilde{g})$ is a HyperFuzzy Trace on $(\mathcal{I}, \mathcal{W}, \mathcal{G})$.

Proof: For each $x \in \mathcal{I}$, $\tilde{v}(x) \neq \varnothing$, so $f_\sharp(\tilde{v}(x))$ is nonempty. Hence $\tilde{v}'$ is well-defined into $\tilde{\mathcal{P}}(\mathcal{W})$, and together with the unchanged gist map $\tilde{g}$ it satisfies the definition of a HyperFuzzy Trace on the new verbatim space $\mathcal{W}$. □

Theorem 3. (Complete Join-Semilattice under Pointwise Union)**.** *The collection of all HyperFuzzy Traces on $(\mathcal{I}, \mathcal{V}, \mathcal{G})$, ordered by*

$$(\tilde{v}_1, \tilde{g}_1) \;\leq\; (\tilde{v}_2, \tilde{g}_2) \quad \Longleftrightarrow \quad \tilde{v}_1(x) \subseteq \tilde{v}_2(x) \text{ and } \tilde{g}_1(x) \subseteq \tilde{g}_2(x)\ \forall x,$$

is a complete join-semilattice. The join of an arbitrary family $\{(\tilde{v}_i, \tilde{g}_i)\}_{i \in I}$ is given by

$$\Big(x \mapsto \bigcup_{i \in I} \tilde{v}_i(x),\ x \mapsto \bigcup_{i \in I} \tilde{g}_i(x) \Big).$$

Proof: For each x, each $\tilde{v}_i(x)$ and $\tilde{g}_i(x)$ is nonempty, so their unions $\bigcup_i \tilde{v}_i(x)$ and $\bigcup_i \tilde{g}_i(x)$ are nonempty. Thus the pointwise union defines valid maps into $\tilde{\mathcal{P}}(\mathcal{V})$ and $\tilde{\mathcal{P}}(\mathcal{G})$. It is evidently the least upper bound under the componentwise inclusion order, making the set of all HyperFuzzy Traces a complete join-semilattice. □

Theorem 4. (Existence of Constant HyperFuzzy Traces)**.** *For any nonempty $A \subseteq \mathcal{V}$ and $B \subseteq \mathcal{G}$, the pair*

$$\tilde{v}(x) = A, \qquad \tilde{g}(x) = B \quad \text{for all } x \in \mathcal{I}$$

defines a HyperFuzzy Trace on $(\mathcal{I}, \mathcal{V}, \mathcal{G})$.

Proof: Since $A \neq \varnothing$ and $B \neq \varnothing$, the constant mappings $x \mapsto A$ and $x \mapsto B$ land in the collections of nonempty subsets of $\mathcal{V}$ and $\mathcal{G}$, respectively. Hence they satisfy the definition of a HyperFuzzy Trace. □

Theorem 5. (Restriction to a Sub-Universe)**.** *If $\mathcal{J} \subseteq \mathcal{I}$, then the restricted maps*

$$\tilde{v}|_{\mathcal{J}} : \mathcal{J} \longrightarrow \tilde{\mathcal{P}}(\mathcal{V}), \quad \tilde{g}|_{\mathcal{J}} : \mathcal{J} \longrightarrow \tilde{\mathcal{P}}(\mathcal{G})$$

define a HyperFuzzy Trace on $(\mathcal{J}, \mathcal{V}, \mathcal{G})$.

Proof: For each $x \in \mathcal{J}$, since $x \in \mathcal{I}$ we have $\tilde{v}(x) \neq \varnothing$ and $\tilde{g}(x) \neq \varnothing$. Hence the restricted maps land in nonempty subsets of $\mathcal{V}$ and $\mathcal{G}$, satisfying the definition of a HyperFuzzy Trace on $\mathcal{J}$. □

Theorem 6. (Recombination of Components). *Let $(\tilde{v}_1, \tilde{g}_1)$ and $(\tilde{v}_2, \tilde{g}_2)$ be HyperFuzzy Traces on $(\mathcal{I}, \mathcal{V}, \mathcal{G})$. Then the pairs*

$$(\tilde{v}_1,\ \tilde{g}_2) \quad \textit{and} \quad (\tilde{v}_2,\ \tilde{g}_1)$$

are also HyperFuzzy Traces on the same universe.

Proof: Each $\tilde{v}_i(x)$ and $\tilde{g}_i(x)$ is nonempty by assumption. Swapping the gist components between the two traces preserves nonemptiness of each map at every $x \in \mathcal{I}$. Therefore each recombined pair still maps x to a nonempty verbatim set and a nonempty gist set, hence defines a valid HyperFuzzy Trace. □

Theorem 7. (Product of HyperFuzzy Traces). *Let $(\tilde{v}_1, \tilde{g}_1)$ be a HyperFuzzy Trace on $(\mathcal{I}, \mathcal{V}_1, \mathcal{G}_1)$ and $(\tilde{v}_2, \tilde{g}_2)$ a HyperFuzzy Trace on $(\mathcal{J}, \mathcal{V}_2, \mathcal{G}_2)$. Define for each $(i, j) \in \mathcal{I} \times \mathcal{J}$,*

$$\tilde{v}((i,j)) = \tilde{v}_1(i) \times \tilde{v}_2(j) \ \subseteq\ \mathcal{V}_1 \times \mathcal{V}_2, \quad \tilde{g}((i,j)) = \tilde{g}_1(i) \times \tilde{g}_2(j) \ \subseteq\ \mathcal{G}_1 \times \mathcal{G}_2.$$

Then $(\tilde{v}, \tilde{g})$ is a HyperFuzzy Trace on $(\mathcal{I} \times \mathcal{J},\ \mathcal{V}_1 \times \mathcal{V}_2,\ \mathcal{G}_1 \times \mathcal{G}_2)$.

Proof: For each (i, j), both $\tilde{v}_1(i)$ and $\tilde{v}_2(j)$ are nonempty, so their Cartesian product is nonempty. Similarly for the gist sets. Hence the pair $(\tilde{v}, \tilde{g})$ assigns each (i, j) nonempty subsets of $\mathcal{V}_1 \times \mathcal{V}_2$ and $\mathcal{G}_1 \times \mathcal{G}_2$, satisfying the HyperFuzzy Trace definition on the product universe. □

Theorem 8. (Pushforward under Verbatim and Gist Maps). *Let $f : \mathcal{V} \to \mathcal{V}'$ and $h : \mathcal{G} \to \mathcal{G}'$ be any functions. Define*

$$f_{\#}(\tilde{v}(x)) = \{\, f(v) \mid v \in \tilde{v}(x)\} \subseteq \mathcal{V}', \quad h_{\#}(\tilde{g}(x)) = \{\, h(g) \mid g \in \tilde{g}(x)\} \subseteq \mathcal{G}'.$$

Then the pair

$$\big(f_{\#} \circ \tilde{v},\ h_{\#} \circ \tilde{g}\big)$$

defines a HyperFuzzy Trace on $(\mathcal{I}, \mathcal{V}', \mathcal{G}')$.

Proof: Since each $\tilde{v}(x) \neq \varnothing$, its image under any function f remains nonempty. The same holds for $\tilde{g}(x)$ under h. Therefore $f_{\#} \circ \tilde{v}$ and $h_{\#} \circ \tilde{g}$ map each $x \in \mathcal{I}$ to nonempty subsets of $\mathcal{V}'$ and $\mathcal{G}'$, satisfying the HyperFuzzy Trace axioms on the new codomains. □

2.3 | SuperHyperFuzzy Trace

A superhyperfuzzy trace assigns hierarchical families of verbatim and gist subsets at multiple powerset levels, modeling nested imprecision.

Definition 9. ((m, n)-SuperHyperFuzzy Trace). Let $\mathcal{I}$ be a set of information items, let $\mathcal{V}$ be the verbatim-encoding space, and let $\mathcal{G} \subseteq [0, 1]$ be the gist-encoding space. Fix nonnegative integers m, n. Recall the nonempty kth powerset

$$\mathcal{P}^*_k(X) = \mathcal{P}\big(\mathcal{P}^*_{k-1}(X)\big) \setminus \{\emptyset\}, \quad \mathcal{P}^*_0(X) = X.$$

An (m, n)-*SuperHyperFuzzy Trace* on $(\mathcal{I}, \mathcal{V}, \mathcal{G})$ is a pair of functions

$$\tilde{v}_m : \mathcal{I} \longrightarrow \mathcal{P}^*_m(\mathcal{V}), \qquad \tilde{g}_n : \mathcal{I} \longrightarrow \mathcal{P}^*_n(\mathcal{G}).$$

For each $x \in \mathcal{I}$ we write

$$\mathrm{SHFT}_{m,n}(x) = \big(\tilde{v}_m(x),\ \tilde{g}_n(x)\big),$$

and call $\mathcal{H}_{m,n} = \{\mathrm{SHFT}_{m,n}(x) \mid x \in \mathcal{I}\}$ the set of all (m, n)-SuperHyperFuzzy Traces.

Example 5 (Real-World $(2,3)$-SuperHyperFuzzy Trace: Eyewitness License-Plate Recall with Confidence Calibration)**.** Let

$$\mathcal{I} = \{\text{"License-Plate Recall"}\}, \quad \mathcal{V} = \{\text{all plausible 7-character alphanumeric plates}\}, \quad \mathcal{G} = [0,1].$$

Fix $m = 2$ and $n = 3$. Then

$$\mathcal{P}_1^*(\mathcal{V}) = \mathcal{P}(\mathcal{V}) \setminus \{\emptyset\}, \quad \mathcal{P}_2^*(\mathcal{V}) = \mathcal{P}\big(\mathcal{P}_1^*(\mathcal{V})\big) \setminus \{\emptyset\},$$

and similarly

$$\begin{aligned}\mathcal{P}_1^*(\mathcal{G}) &= \{A \subseteq [0,1] \mid A \neq \emptyset\},\\ \mathcal{P}_2^*(\mathcal{G}) &= \mathcal{P}(\mathcal{P}_1^*(\mathcal{G})) \setminus \{\emptyset\},\\ \mathcal{P}_3^*(\mathcal{G}) &= \mathcal{P}(\mathcal{P}_2^*(\mathcal{G})) \setminus \{\emptyset\}.\end{aligned}$$

We model two rounds of verbatim recall and three hierarchical rounds of confidence judgment:

Verbatim super-set $\tilde{v}_2$:

$$\tilde{v}_2\big(\text{"License-Plate Recall"}\big) = \Big\{\underbrace{\{\text{"ABC1234", "ABC1E34"}\}}_{\text{Initial interview guesses}}, \underbrace{\{\text{"ABC1234", "ABD1234"}\}}_{\text{Follow-up guesses}}\Big\} \subseteq \mathcal{P}_2^*(\mathcal{V}).$$

Gist super-set $\tilde{g}_3$:

$$\tilde{g}_3\big(\text{"License-Plate Recall"}\big) = \Big\{\underbrace{\{\{0.50,0.60\}, \{0.55,0.65\}\}}_{\substack{\text{Self-rated con}\tilde{}\text{dence}\\ \text{immediately}}}, \underbrace{\{\{0.70,0.80\}, \{0.75,0.85\}\}}_{\text{After photo lineup}}\Big\} \subseteq \mathcal{P}_3^*(\mathcal{G}).$$

Here each inner pair $\{0.50,0.60\}, \{0.55,0.65\}$ lies in $\mathcal{P}_1^*(\mathcal{G})$, their collection lies in $\mathcal{P}_2^*(\mathcal{G})$, and the outer braces yield an element of $\mathcal{P}_3^*(\mathcal{G})$.

Hence the $(2,3)$-SuperHyperFuzzy Trace is

$$\mathrm{SHFT}_{2,3}\big(\text{"License-Plate Recall"}\big) = \Big(\tilde{v}_2(\text{"License-Plate Recall"}),\ \tilde{g}_3(\text{"License-Plate Recall"})\Big).$$

This example encodes two levels of verbatim uncertainty (initial vs. follow-up) and three nested levels of confidence uncertainty (immediate, post-lineup, expert-calibrated).

Example 6 (Real-World $(3,3)$-SuperHyperFuzzy Trace: ATM PIN Multi-Session Recall)**.** Let

$$\mathcal{I} = \{\text{"ATM PIN Recall"}\}, \quad \mathcal{V} = \{\text{all plausible 4-digit PIN codes}\}, \quad \mathcal{G} = [0,1].$$

Fix $m = 3$ and $n = 3$. Then the nonempty iterated powersets are defined as above. We model three recall sessions and three stages of confidence calibration:

Verbatim super-set $\tilde{v}_3$:

$$\begin{aligned}\tilde{v}_3\big(\text{"ATM PIN Recall"}\big) = \Big\{&\underbrace{\{\{\text{"1234", "1243"}\}, \{\text{"1234", "1324"}\}\}}_{\text{Session 1 guesses}},\\ &\underbrace{\{\{\text{"1234", "1235"}\}, \{\text{"1234", "2134"}\}\}}_{\text{Session 2 guesses}}\Big\}\\ &\subseteq \mathcal{P}_3^*(\mathcal{V}).\end{aligned}$$

Each inner set of codes lies in $\mathcal{P}_1^*(\mathcal{V})$; each pair of such lies in $\mathcal{P}_2^*(\mathcal{V})$; and the outer collection yields an element of $\mathcal{P}_3^*(\mathcal{V})$.

Gist super-set $\tilde{g}_3$:

$$\tilde{g}_3\big(\text{"ATM PIN Recall"}\big) = \Big\{\underbrace{\{\{0.40,0.50\}, \{0.45,0.55\}\}}_{\text{Immediate con}\tilde{}\text{dence}}, \underbrace{\{\{0.65,0.75\}, \{0.68,0.78\}\}}_{\text{After hint prompt}}\Big\} \subseteq \mathcal{P}_3^*(\mathcal{G}).$$

Each inner pair of confidence values lies in $\mathcal{P}_1^*(\mathcal{G})$, their families in $\mathcal{P}_2^*(\mathcal{G})$, and the outer braces in $\mathcal{P}_3^*(\mathcal{G})$.

Thus the $(3,3)$-SuperHyperFuzzy Trace is

$$\mathrm{SHFT}_{3,3}\big(\text{"ATM PIN Recall"}\big) = \Big(\tilde{v}_3(\text{"ATM PIN Recall"}),\ \tilde{g}_3(\text{"ATM PIN Recall"})\Big).$$

This captures three hierarchical levels of verbatim recall (three sessions) and three nested levels of confidence (immediate, post-hint, final calibration) in an ATM PIN memory task.

Example 7. (Real-World $(2,2)$-SuperHyperFuzzy Trace: Eyewitness License-Plate Recall)**.** Let

$$\mathcal{I} = \{\text{"License-Plate Recall"}\}, \quad \mathcal{V} = \{\text{all plausible 7-character alphanumeric strings}\}, \quad \mathcal{G} = [0,1].$$

Fix $m = n = 2$. Then

$$\mathcal{P}_1^*(\mathcal{V}) = \mathcal{P}(\mathcal{V}) \setminus \{\emptyset\}, \quad \mathcal{P}_2^*(\mathcal{V}) = \mathcal{P}\big(\mathcal{P}_1^*(\mathcal{V})\big) \setminus \{\emptyset\},$$

and similarly for $\mathcal{G}$. We model two rounds of recall and two rounds of confidence judgments:

Verbatim super-set $\tilde{v}_2$:

$$\tilde{v}_2\big(\text{"License-Plate Recall"}\big) = \Big\{\underbrace{\{\text{"ABC1234"}, \text{"ABC1E34"}\}}_{\substack{\text{Initial interview:}\\ \text{participant's two best guesses}}}, \underbrace{\{\text{"ABC1234"}, \text{"ABD1234"}\}}_{\substack{\text{Follow-up interview:}\\ \text{remembered with slight variation}}}\Big\} \subseteq \mathcal{P}_2^*(\mathcal{V}).$$

Gist super-set $\tilde{g}_2$:

$$\tilde{g}_2\big(\text{"License-Plate Recall"}\big) = \Big\{ \underbrace{\{0.6,\ 0.7\}}_{\substack{\text{Self-rated confidence}\\ \text{in initial recall}}}, \underbrace{\{0.8,\ 0.9\}}_{\substack{\text{Self-rated confidence}\\ \text{after seeing photo lineup}}} \Big\} \subseteq \mathcal{P}_2^*(\mathcal{G}).$$

Hence the $(2,2)$-SuperHyperFuzzy Trace of the single item is

$$\mathrm{SHFT}_{2,2}\big(\text{"License-Plate Recall"}\big) = \Big(\tilde{v}_2(\text{"License-Plate Recall"}),\ \tilde{g}_2(\text{"License-Plate Recall"})\Big).$$

This example captures two hierarchical levels of verbatim uncertainty (initial vs. follow-up guesses) and two levels of gist uncertainty (initial vs. confirmed confidence) in an eyewitness memory task.

Example 8. (Real-World $(2,2)$-SuperHyperFuzzy Trace: Air Quality Assessment)**.** Let

$$\mathcal{I} = \{\text{“AQI reading”}\}$$

be the item “Air Quality Index reading.” Define the *verbatim-encoding space*

$$\mathcal{V} = \{\,\text{integer AQI values from 0 to 500}\},$$

and the *gist-encoding space*

$$\mathcal{G} \subseteq [0,1]$$

where each $g \in \mathcal{G}$ represents a normalized pollution risk score.

We fix $m = 2$ and $n = 2$. Then

$$\mathcal{P}_1^*(\mathcal{V}) = \mathcal{P}(\mathcal{V}) \setminus \{\emptyset\}, \quad \mathcal{P}_2^*(\mathcal{V}) = \mathcal{P}\big(\mathcal{P}_1^*(\mathcal{V})\big) \setminus \{\emptyset\},$$

and similarly for $\mathcal{G}$.

We model two layers of measurement uncertainty:

Verbatim hyper-set $\tilde{v}_2$:

$$\tilde{v}_2(\text{“AQI reading”}) = \Big\{ \underbrace{\{50, 55, 60\}}_{\substack{\text{Sensor group A's}\\ \text{daily readings}}}, \underbrace{\{52, 58\}}_{\substack{\text{Sensor group B's}\\ \text{daily readings}}} \Big\} \subseteq \mathcal{P}_1^*(\mathcal{V}) \subseteq \mathcal{P}_2^*(\mathcal{V}).$$

Here each inner set is a nonempty subset of $\mathcal{V}$, and the outer braces form a nonempty family of such subsets.

Gist hyper-set $\tilde{g}_2$:

$$\tilde{g}_2(\text{“AQI reading”}) = \Big\{ \underbrace{\{0.1, 0.12\}}_{\substack{\text{Expert panel's}\\ \text{risk scores}}}, \underbrace{\{0.08, 0.11, 0.13\}}_{\substack{\text{Automated model's}\\ \text{risk scores}}} \Big\} \subseteq \mathcal{P}_1^*(\mathcal{G}) \subseteq \mathcal{P}_2^*(\mathcal{G}).$$

Each inner set is a nonempty subset of $\mathcal{G}$, and the outer set is a nonempty collection of these.

Thus the $(2,2)$-SuperHyperFuzzy Trace is

$$\mathrm{SHFT}_{2,2}\big(\text{“AQI reading”}\big) = \big(\tilde{v}_2(\text{“AQI reading”}),\, \tilde{g}_2(\text{“AQI reading”})\big).$$

This concrete example illustrates two hierarchical levels of uncertainty in both the exact AQI measurements (sensor groups A and B) and the corresponding risk scores (expert vs. automated), encoded as a valid $(2,2)$-SuperHyperFuzzy Trace.

Theorem 9. *Let $\big(\tilde{v}_m, \tilde{g}_n\big)$ be an (m,n)-SuperHyperFuzzy Trace. Then:*

(1) Generalization of Fuzzy Trace: *If $m = n = 0$, then $\mathcal{P}_0^*(\mathcal{V}) = \mathcal{V}$ and $\mathcal{P}_0^*(\mathcal{G}) = \mathcal{G}$, so $\tilde{v}_0 : \mathcal{I} \to \mathcal{V}$ and $\tilde{g}_0 : \mathcal{I} \to \mathcal{G}$. Setting $v(x) = \tilde{v}_0(x)$ and $g(x) = \tilde{g}_0(x)$ recovers exactly the classical fuzzy trace $x \mapsto (v(x), g(x))$.*

(2) Generalization of HyperFuzzy Trace: *If $m = n = 1$, then $\mathcal{P}_1^*(\mathcal{V}) = \tilde{\mathcal{P}}(\mathcal{V})$ and $\mathcal{P}_1^*(\mathcal{G}) = \tilde{\mathcal{P}}(\mathcal{G})$. Hence $\tilde{v}_1 : \mathcal{I} \to \tilde{\mathcal{P}}(\mathcal{V})$ and $\tilde{g}_1 : \mathcal{I} \to \tilde{\mathcal{P}}(\mathcal{G})$ define a HyperFuzzy Trace.*

(3) (m,n)-SuperHyperFuzzy Set Structure: *Define the canonical embedding*

$$\iota_m \colon \mathcal{I} \longrightarrow \mathcal{P}_m^*(\mathcal{I}), \quad \iota_0(x) = x, \quad \iota_{k+1}(x) = \{\iota_k(x)\}.$$

Then by composing with the gist mapping we obtain

$$\tilde{\mu}_{m,n} = \tilde{g}_n \circ \iota_m : \mathcal{P}_m^*(\mathcal{I}) \longrightarrow \mathcal{P}_n^*(\mathcal{G}),$$

which is exactly the definition of an (m,n)-SuperHyperFuzzy Set on the universe $\mathcal{I}$ with codomain $\mathcal{G}$. The same construction applies to $\tilde{v}_m$, yielding an (m,n)-SuperHyperFuzzy Set into $\mathcal{V}$.

Proof: (1) When $m = n = 0$, by definition $\mathcal{P}_0^*(X) = X$. Thus $\tilde{v}_0$ and $\tilde{g}_0$ are ordinary single-valued functions into $\mathcal{V}$ and $\mathcal{G}$. Their pairing is precisely a fuzzy trace.

(2) When $m = n = 1$, we have $\mathcal{P}_1^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$. Hence $\tilde{v}_1$ and $\tilde{g}_1$ assign each x a nonempty subset of $\mathcal{V}$ or $\mathcal{G}$, exactly the data of a HyperFuzzy Trace.

(3) To exhibit the SuperHyperFuzzy Set structure, note that the embedding ι_m sends each item x into the mth nonempty powerset of $\mathcal{I}$. Composition $\tilde{\mu}_{m,n} = \tilde{g}_n \circ \iota_m$ therefore defines a function $\mathcal{P}_m^*(\mathcal{I}) \to \mathcal{P}_n^*(\mathcal{G})$. By Definition 1.7, any such mapping is an (m,n)-SuperHyperFuzzy Set. An entirely analogous argument applies to $\tilde{v}_m$.

□

Theorem 10. (Reduction to Fuzzy Trace)**.** *Let $\big(\tilde{v}_m, \tilde{g}_n\big)$ be an (m,n)-SuperHyperFuzzy Trace on $(\mathcal{I}, \mathcal{V}, \mathcal{G})$. If $m = n = 0$, then $\mathcal{P}_0^*(\mathcal{V}) = \mathcal{V}$ and $\mathcal{P}_0^*(\mathcal{G}) = \mathcal{G}$, so*

$$\tilde{v}_0 : \mathcal{I} \longrightarrow \mathcal{V}, \quad \tilde{g}_0 : \mathcal{I} \longrightarrow \mathcal{G}.$$

Setting

$$v(x) = \tilde{v}_0(x), \quad g(x) = \tilde{g}_0(x),$$

one recovers exactly a classical fuzzy trace $x \mapsto \big(v(x), g(x)\big)$.

Proof: By definition of the nonempty powerset, $\mathcal{P}_0^*(X) = X$ for any set X. Hence when $m = n = 0$, $\tilde{v}_0$ and $\tilde{g}_0$ are ordinary single-valued functions $\mathcal{I} \to \mathcal{V}$ and $\mathcal{I} \to \mathcal{G}$. Their pairing $\mathrm{FT}(x) = (v(x), g(x))$ satisfies the definition of a fuzzy trace. □

Theorem 11. (Reduction to HyperFuzzy Trace). *Let* $\left(\tilde{v}_m, \tilde{g}_n\right)$ *be an* (m,n)*-SuperHyperFuzzy Trace. If* $m = n = 1$*, then* $\mathcal{P}_1^*(X) = \mathcal{P}(X) \setminus \{\emptyset\} = \tilde{\mathcal{P}}(X)$*. Thus*

$$\tilde{v}_1 : \mathcal{I} \longrightarrow \tilde{\mathcal{P}}(\mathcal{V}), \quad \tilde{g}_1 : \mathcal{I} \longrightarrow \tilde{\mathcal{P}}(\mathcal{G})$$

defines exactly a HyperFuzzy Trace on $(\mathcal{I}, \mathcal{V}, \mathcal{G})$*.*

Proof: When $m = n = 1$, by definition $\mathcal{P}_1^*(X)$ is the collection of all nonempty subsets of X. Therefore $\tilde{v}_1$ and $\tilde{g}_1$ assign each $x \in \mathcal{I}$ a nonempty subset of $\mathcal{V}$ resp. $\mathcal{G}$, which is precisely the data of a HyperFuzzy Trace. □

Theorem 12. (Level-Projection Monotonicity). *Let* $\left(\tilde{v}_m, \tilde{g}_n\right)$ *be an* (m,n)*-SuperHyperFuzzy Trace. Fix integers* $0 \le k \le m$ *and* $0 \le \ell \le n$*. Define the* flattening map

$$\varphi^{(1)} : \mathcal{P}_{r+1}^*(X) \longrightarrow \mathcal{P}_r^*(X), \quad \varphi^{(1)}(A) = \bigcup_{B \in A} B,$$

and for $r + 1 > k$ *iterate* $\varphi^{(r)} = \varphi^{(1)} \circ \varphi^{(r-1)}$*. Then*

$$\tilde{v}_k = \varphi^{(m-k)} \circ \tilde{v}_m : \mathcal{I} \longrightarrow \mathcal{P}_k^*(\mathcal{V}),$$

$$\tilde{g}_\ell = \varphi^{(n-\ell)} \circ \tilde{g}_n : \mathcal{I} \longrightarrow \mathcal{P}_\ell^*(\mathcal{G}).$$

The pair $\left(\tilde{v}_k, \tilde{g}_\ell\right)$ *is then a valid* (k,ℓ)*-SuperHyperFuzzy Trace on* $(\mathcal{I}, \mathcal{V}, \mathcal{G})$*.*

Proof: We must show two things:
(i) Well-definedness of $\tilde{v}_k$. Since $\tilde{v}_m(x) \in \mathcal{P}_m^*(\mathcal{V})$, it is a nonempty family of elements of $\mathcal{P}_{m-1}^*(\mathcal{V})$. Applying $\varphi^{(1)}$ once yields $\varphi^{(1)}(\tilde{v}_m(x)) \in \mathcal{P}_{m-1}^*(\mathcal{V})$. Iterating $m - k$ times,

$$\varphi^{(m-k)}(\tilde{v}_m(x)) \in \mathcal{P}_k^*(\mathcal{V}),$$

and remains nonempty at each step because the union of nonempty sets is nonempty. Thus $\tilde{v}_k(x)$ is well-defined as an element of $\mathcal{P}_k^*(\mathcal{V})$. The same argument applies to $\tilde{g}_\ell$.
(ii) Preservation of trace structure. By construction, $\tilde{v}_k : \mathcal{I} \to \mathcal{P}_k^*(\mathcal{V})$ and $\tilde{g}_\ell : \mathcal{I} \to \mathcal{P}_\ell^*(\mathcal{G})$. Hence $\mathrm{SHFT}_{k,\ell}(x) = (\tilde{v}_k(x), \tilde{g}_\ell(x))$ satisfies the definition of an (k,ℓ)-SuperHyperFuzzy Trace. □

3 | Result: Fuzzy Front-End

3.1 | Fuzzy Front-End

A Fuzzy Front-End uses graded membership functions to evaluate and aggregate multiple criteria, enabling soft go/no-go decisions on innovation ideas (cf.[25, 26, 27]).

Definition 10. (Fuzzy Front-End of Innovation). (cf.[28, 29, 30]) Let $\mathcal{I}$ be the universe of potential project ideas and let

$$\mathcal{C} = \{c_1, c_2, \ldots, c_m\}$$

be a finite set of evaluation criteria. A *Fuzzy Front-End* (FFE) is the triple

$$\big(\mathcal{I},\ \mathcal{C},\ \mu\big),$$

where

$$\mu : \mathcal{I} \times \mathcal{C} \longrightarrow [0,1]$$

is a *membership function* assigning to each idea $i \in \mathcal{I}$ and criterion c_j a degree

$$\mu(i, c_j) = \text{“satisfaction of criterion } c_j \text{ by idea } i\text{”} \in [0,1].$$

Moreover, fix a fuzzy aggregation operator

$$F : [0,1]^m \longrightarrow [0,1],$$

for example a weighted average or a fuzzy min–max operator. Then the overall readiness of i is

$$r(i) = F\big(\mu(i,c_1),\ \mu(i,c_2),\ \ldots,\ \mu(i,c_m)\big).$$

Finally, for a given threshold $\alpha \in [0,1]$, the transition to formal development occurs exactly for those ideas satisfying

$$r(i) \;\geq\; \alpha.$$

Remark 2. In practice one often takes

$$F(x_1,\ldots,x_m) = \sum_{j=1}^{m} w_j\, x_j, \qquad \sum_{j=1}^{m} w_j = 1,$$

or the fuzzy max–min operator

$$F(x_1,\ldots,x_m) = \min_j \big\{\max\{w_j,\, x_j\}\big\},$$

with weights $w_j \in [0,1]$. This captures the idea that an FFE decision is inherently fuzzy, blending multiple graded criteria before a crisp go/no-go cutoff.

Example 9. (Fuzzy Front-End: Screening New Product Ideas)**.** Let

$$\mathcal{I} = \{\text{AI Tutor Platform},\ \text{Eco-Friendly Packaging},\ \text{IoT Soil Sensor}\},$$

and let the evaluation criteria be

$$\mathcal{C} = \{\text{Feasibility},\ \text{Market Demand},\ \text{Resource Efficiency}\}.$$

We define the membership function $\mu : \mathcal{I} \times \mathcal{C} \to [0,1]$ by the following table of scores:

$\mu(i,c)$ =

	Feasibility	Market Demand	Resource Efficiency
AI Tutor Platform	0.6	0.9	0.4
Eco-Friendly Packaging	0.8	0.7	0.6
IoT Soil Sensor	0.7	0.8	0.5

Choose the weighted-average aggregator

$$F(x,y,z) \;=\; 0.4x \;+\; 0.4y \;+\; 0.2z.$$

Then the overall readiness scores $r(i) = F(\mu(i,c_1), \mu(i,c_2), \mu(i,c_3))$ are:

$$\begin{aligned} r(\text{AI Tutor Platform}) &= 0.4\cdot 0.6 + 0.4\cdot 0.9 + 0.2\cdot 0.4 = 0.68,\\ r(\text{Eco-Friendly Packaging}) &= 0.4\cdot 0.8 + 0.4\cdot 0.7 + 0.2\cdot 0.6 = 0.72,\\ r(\text{IoT Soil Sensor}) &= 0.4\cdot 0.7 + 0.4\cdot 0.8 + 0.2\cdot 0.5 = 0.70. \end{aligned}$$

For a development threshold $\alpha = 0.70$, both Eco-Friendly Packaging (0.72) and IoT Soil Sensor (0.70) advance to formal development, while the AI Tutor Platform (0.68) is deferred.

This example illustrates how a Fuzzy Front-End uses graded membership and aggregation to screen innovation ideas under multiple, possibly competing criteria.

3.2 | HyperFuzzy Front

A HyperFuzzy Front-End extends the FFE by mapping each idea-criterion pair to sets of plausible membership degrees, capturing expert variability.

Definition 11. (HyperFuzzy Front-End of Innovation)**.** Let $\mathcal{I}$ be the universe of potential project ideas and let

$$\mathcal{C} = \{c_1, c_2, \ldots, c_m\}$$

be a finite set of evaluation criteria. Denote by

$$\tilde{\mathcal{P}}([0,1]) = \{\, A \subseteq [0,1] \mid A \neq \varnothing\}$$

the collection of all nonempty subsets of $[0,1]$. A HyperFuzzy Front-End (HFFE) is the triple

$$\big(\mathcal{I},\ \mathcal{C},\ \tilde{\mu}\big),$$

where

$$\tilde{\mu} : \mathcal{I} \times \mathcal{C} \longrightarrow \tilde{\mathcal{P}}([0,1])$$

is a *hyper-membership function* assigning to each idea $i \in \mathcal{I}$ and criterion c_j a nonempty set

$$\tilde{\mu}(i, c_j) = \{\text{degrees to which } i \text{ satisfies } c_j\} \subseteq [0,1].$$

Moreover, let

$$F : [0,1]^m \longrightarrow [0,1]$$

be a classical fuzzy aggregation operator (e.g. weighted average or min–max). Define the induced *hyper-aggregation operator*

$$\widetilde{F} : \big(\tilde{\mathcal{P}}([0,1])\big)^m \longrightarrow \tilde{\mathcal{P}}([0,1]),$$
$$\widetilde{F}(A_1, \ldots, A_m) = \{\, F(x_1, \ldots, x_m) \mid x_j \in A_j \}.$$

Then the *readiness hyper-set* of an idea i is

$$\tilde{r}(i) = \widetilde{F}\big(\tilde{\mu}(i, c_1), \ldots, \tilde{\mu}(i, c_m)\big) \subseteq [0,1].$$

Finally, for a threshold $\alpha \in [0,1]$, idea i "passes" if and only if

$$\tilde{r}(i) \cap [\alpha, 1] \neq \varnothing.$$

Example 10. (HyperFuzzy Front-End: Screening New Product Ideas under Expert Variability)**.** Let

$$\mathcal{I} = \{\text{AI Tutor Platform}, \text{ Eco-Friendly Packaging}, \text{ IoT Soil Sensor}\},$$

and let the evaluation criteria be

$$\mathcal{C} = \{\text{Feasibility}, \text{ Market Demand}, \text{ Resource Efficiency}\}.$$

We model expert disagreement by assigning each (i, c_j) a hyper-membership set of plausible satisfaction degrees:

$$\tilde{\mu}(i, c_j) \subseteq [0,1].$$

For instance:

AI Tutor Platform:

$$\tilde{\mu}(\text{AI Tutor}, \text{Feasibility}) = \{0.5, 0.6\},$$
$$\tilde{\mu}(\text{AI Tutor}, \text{Market Demand}) = \{0.8, 0.9\},$$
$$\tilde{\mu}(\text{AI Tutor}, \text{Resource Efficiency}) = \{0.3, 0.4\}.$$

Eco-Friendly Packaging:

$$\tilde{\mu}(\text{Eco Packaging}, \text{Feasibility}) = \{0.7, 0.8\},$$
$$\tilde{\mu}(\text{Eco Packaging}, \text{Market Demand}) = \{0.6, 0.7\},$$
$$\tilde{\mu}(\text{Eco Packaging}, \text{Resource Efficiency}) = \{0.5, 0.6\}.$$

IoT Soil Sensor:

$$\tilde{\mu}(\text{IoT Sensor}, \text{Feasibility}) = \{0.6, 0.7\},$$
$$\tilde{\mu}(\text{IoT Sensor}, \text{Market Demand}) = \{0.7, 0.8\},$$
$$\tilde{\mu}(\text{IoT Sensor}, \text{Resource Efficiency}) = \{0.4, 0.5\}.$$

Choose the weighted-average aggregator

$$F(x, y, z) = 0.4\,x + 0.4\,y + 0.2\,z.$$

The induced hyper-aggregation operator is

$$\widetilde{F}(A, B, C) = \{\, F(x, y, z) \mid x \in A,\ y \in B,\ z \in C \} \subseteq [0,1].$$

Thus, for each idea i the readiness hyper-set is

$$\tilde{r}(i) = \widetilde{F}\big(\tilde{\mu}(i, \text{Feasibility}),\ \tilde{\mu}(i, \text{Market Demand}),\ \tilde{\mu}(i, \text{Resource Efficiency})\big).$$

Concretely:

$$\begin{aligned}\tilde{r}(\text{AI Tutor}) &= \widetilde{F}(\{0.5,0.6\},\{0.8,0.9\},\{0.3,0.4\})\\ &= \big\{\, 0.4x+0.4y+0.2z \mid x\in\{0.5,0.6\},\ y\in\{0.8,0.9\},\ z\in\{0.3,0.4\}\big\}\\ &= \{0.59,0.63,0.65,0.69\},\end{aligned}$$

$$\begin{aligned}\tilde{r}(\text{Eco-Packaging}) &= \widetilde{F}(\{0.7,0.8\},\{0.6,0.7\},\{0.5,0.6\})\\ &= \{0.62,0.66,0.68,0.72\},\end{aligned}$$

$$\begin{aligned}\tilde{r}(\text{IoT Sensor}) &= \widetilde{F}(\{0.6,0.7\},\{0.7,0.8\},\{0.4,0.5\})\\ &= \{0.60,0.64,0.66,0.70\}.\end{aligned}$$

For a threshold $\alpha = 0.65$, an idea "passes" if its hyper-set intersects $[0.65, 1]$. Hence:

$$\tilde{r}(\text{AI Tutor}) \cap [0.65,1] = \{0.65,0.69\} \neq \varnothing,$$

$$\tilde{r}(\text{Eco-Packaging}) \cap [0.65,1] = \{0.66,0.68,0.72\} \neq \varnothing,$$

$$\tilde{r}(\text{IoT Sensor}) \cap [0.65,1] = \{0.66,0.70\} \neq \varnothing.$$

All three ideas advance to formal development under this HyperFuzzy Front-End evaluation.

Theorem 13. *Let $(\mathcal{I},\mathcal{C},\tilde{\mu})$ be a HyperFuzzy Front-End as above. Then:*

(1) ***Generalization of Fuzzy Front-End.*** *If for every (i,c_j) the hyper-membership set $\tilde{\mu}(i,c_j)$ is a singleton $\{\mu_0(i,c_j)\} \subseteq [0,1]$, then $\tilde{\mu}$ reduces to an ordinary membership function $\mu_0 : \mathcal{I}\times\mathcal{C} \to [0,1]$, and $\tilde{r}(i) = \{r_0(i)\}$ where $r_0(i) = F\big(\mu_0(i,c_1),\dots,\mu_0(i,c_m)\big)$ is the classical readiness score.*

(2) ***HyperFuzzy Set Structure.*** *For each fixed criterion $c_j \in \mathcal{C}$, the map*

$$\tilde{\mu}_{c_j} : \mathcal{I} \longrightarrow \tilde{\mathcal{P}}([0,1]), \quad i \mapsto \tilde{\mu}(i,c_j),$$

assigns to each idea a nonempty subset of $[0,1]$, hence is exactly a HyperFuzzy Set on the universe $\mathcal{I}$ [11, 14].

Proof: (1) If $\tilde{\mu}(i,c_j) = \{\mu_0(i,c_j)\}$ for all i,j, define $\mu_0(i,c_j)$ accordingly. Then

$$\widetilde{F}\big(\{\mu_0(i,c_1)\},\dots,\{\mu_0(i,c_m)\}\big) = \{F(\mu_0(i,c_1),\dots,\mu_0(i,c_m))\} = \{r_0(i)\},$$

recovering the classical Fuzzy Front-End.

(2) By construction $\tilde{\mu}_{c_j}(i) \subseteq [0,1]$ is nonempty for every $i \in \mathcal{I}$. This is precisely the definition of a HyperFuzzy Set on $\mathcal{I}$ with membership mapping into $\tilde{\mathcal{P}}([0,1])$.

□

Theorem 14. (Monotonicity under Non-decreasing Aggregator)**.** *Let $F : [0,1]^m \to [0,1]$ be non-decreasing in each argument. If for two HyperFuzzy Front-Ends $\tilde{\mu}$ and $\tilde{\nu}$ one has*

$$\tilde{\mu}(i,c_j) \ \subseteq\ \tilde{\nu}(i,c_j) \quad \text{for all } i\in\mathcal{I},\ j=1,\dots,m,$$

then their readiness hyper-sets satisfy

$$\widetilde{F}\big(\tilde{\mu}(i,c_1),\dots,\tilde{\mu}(i,c_m)\big) \ \subseteq\ \widetilde{F}\big(\tilde{\nu}(i,c_1),\dots,\tilde{\nu}(i,c_m)\big).$$

Proof: By definition,

$$\widetilde{F}(A_1,\dots,A_m) = \{\, F(x_1,\dots,x_m) \mid x_j \in A_j\}.$$

If $A_j \subseteq B_j$ for each j, then every tuple $(x_1,\dots,x_m)$ with $x_j \in A_j$ also lies in $B_1\times\cdots\times B_m$. Since F is non-decreasing, each value $F(x_1,\dots,x_m)$ computed from A_j-values is also obtained (or exceeded) from B_j-values. Hence

$$\widetilde{F}(A_1,\dots,A_m) \subseteq \widetilde{F}(B_1,\dots,B_m).$$

□

Theorem 17. (Idempotence of Hyper-Aggregation). *Suppose F satisfies $F(x,\dots,x)=x$ for all $x\in[0,1]$. Then for any nonempty $A\subseteq[0,1]$,*

$$\widetilde{F}(A,\dots,A)=\{\,F(x_1,\dots,x_m)\mid x_j\in A\}=A.$$

Proof: Since each $x_j\in A$, and F is idempotent,

$$\{\,F(x_1,\dots,x_m)\mid x_j\in A\}=\{\,x\mid x\in A\}=A.$$

Thus hyper-aggregating the same set across all criteria returns the original set. □

Theorem 18. (Symmetry under Permutation). *If F is symmetric (i.e. $F(x_1,\dots,x_m)=F(x_{\pi(1)},\dots,x_{\pi(m)})$ for every permutation π), then for any permutation π of $\{1,\dots,m\}$ and any nonempty $A_1,\dots,A_m\subseteq[0,1]$,*

$$\widetilde{F}(A_1,\dots,A_m)=\widetilde{F}\big(A_{\pi(1)},\dots,A_{\pi(m)}\big).$$

Proof: By definition,

$$\widetilde{F}(A_1,\dots,A_m)=\{\,F(x_1,\dots,x_m)\mid x_j\in A_j\}.$$

Under the permutation π, the right–hand side becomes

$$\{\,F(x_{\pi(1)},\dots,x_{\pi(m)})\mid x_{\pi(j)}\in A_{\pi(j)}\}.$$

Since F is symmetric, each value $F(x_1,\dots,x_m)$ equals $F(x_{\pi(1)},\dots,x_{\pi(m)})$, and the indexing of sets does not change the collection of tuples. Hence the two hyper-aggregation results coincide. □

Theorem 19. (Bounds of the Readiness Hyper-Set). *Assume F is non-decreasing in each argument. For each idea $i\in\mathcal{I}$ and each criterion c_j, set*

$$a_j=\inf\tilde{\mu}(i,c_j),\quad b_j=\sup\tilde{\mu}(i,c_j).$$

Then the readiness hyper-set

$$\tilde{r}(i)=\widetilde{F}\big(\tilde{\mu}(i,c_1),\dots,\tilde{\mu}(i,c_m)\big)$$

satisfies

$$F(a_1,\dots,a_m)=\inf\tilde{r}(i),\quad F(b_1,\dots,b_m)=\sup\tilde{r}(i).$$

Proof: Because F is non-decreasing, for any choice $x_j\in\tilde{\mu}(i,c_j)$ we have

$$F(a_1,\dots,a_m)\ \le\ F(x_1,\dots,x_m)\ \le\ F(b_1,\dots,b_m).$$

Taking the infimum over all such tuples yields $\inf\tilde{r}(i)=F(a_1,\dots,a_m)$, and taking the supremum yields $\sup\tilde{r}(i)=F(b_1,\dots,b_m)$. □

Theorem 20. (HyperFuzzy Structure of Readiness). *The mapping*

$$\tilde{r}:\mathcal{I}\ \longrightarrow\ \tilde{\mathcal{P}}([0,1]),\quad i\mapsto\tilde{r}(i)$$

is itself a HyperFuzzy Set on the universe $\mathcal{I}$.

Proof: By construction $\tilde{r}(i)\subseteq[0,1]$ is nonempty for each i, so $\tilde{r}:\mathcal{I}\to\tilde{\mathcal{P}}([0,1])$. This exactly matches the definition of a HyperFuzzy Set on $\mathcal{I}$. □

Theorem 21. (Minkowski-Sum Representation for Linear Aggregators). *Suppose*

$$F(x_1,\dots,x_m)=\sum_{j=1}^{m}w_j\,x_j,\quad \sum_{j=1}^{m}w_j=1,\quad w_j\ge0.$$

Then for any nonempty $A_1,\dots,A_m\subseteq[0,1]$,

$$\widetilde{F}(A_1,\dots,A_m)=\big\{\,w_1x_1+\cdots+w_mx_m\mid x_j\in A_j\big\},$$

which is the weighted Minkowski sum *of the sets A_j.*

Proof: Directly from the definition of $\widetilde{F}$ and the linear form of F:

$$\widetilde{F}(A_1, \dots, A_m) = \{ F(x_1, \dots, x_m) \mid x_j \in A_j \} = \{ w_1 x_1 + \cdots + w_m x_m \mid x_j \in A_j \},$$

which by definition is the weighted Minkowski sum. □

3.3 | SuperHyperFuzzy Front-End

A SuperHyperFuzzy Front-End assigns hierarchical families of membership-degree sets to each idea and criterion, modeling nested uncertainty and aggregated readiness.

Definition 12. ((m,n)-SuperHyperFuzzy Front-End)**.** Let $\mathcal{I}$ be the universe of project ideas and $\mathcal{C} = \{c_1, \dots, c_m\}$ a finite set of evaluation criteria. Fix nonnegative integers m, n. Recall the nonempty kth powerset

$$\mathcal{P}_k^*(X) = \mathcal{P}\big(\mathcal{P}_{k-1}^*(X)\big) \setminus \{\emptyset\}, \quad \mathcal{P}_0^*(X) = X.$$

Denote by $\tilde{\mathcal{P}}_n^*([0,1]) = \mathcal{P}_n^*([0,1])$ the family of all nonempty subsets of the nth iterated powerset of $[0,1]$. An (m,n)-*SuperHyperFuzzy Front-End* (SHFFE) is the quadruple

$$\big(\mathcal{I},\ \mathcal{C},\ \tilde{\mu}_{m,n},\ F\big),$$

where

$$\tilde{\mu}_{m,n} : \mathcal{I} \times \mathcal{C} \ \longrightarrow\ \mathcal{P}_n^*([0,1]), \quad (i, c_j) \mapsto \tilde{\mu}_{m,n}(i, c_j),$$

assigns each idea–criterion pair a nonempty family of membership-degree sets in the nth powerset of $[0,1]$, and

$$F : [0,1]^m \ \longrightarrow\ [0,1]$$

is a classical fuzzy aggregator. Define the induced *superhyper-aggregation*

$$\widetilde{F}_{m,n} : \big(\mathcal{P}_n^*([0,1])\big)^m \ \longrightarrow\ \mathcal{P}_n^*([0,1]),$$

$$\widetilde{F}_{m,n}(A_1, \dots, A_m) = \big\{ F(x_1, \dots, x_m) \mid x_j \in A_j \big\}_n^*,$$

where $\{\cdots\}_n^*$ denotes the nonempty nth-powerset closure. Then the *readiness superhyper-set* of idea i is

$$\widetilde{r}_{m,n}(i) = \widetilde{F}_{m,n}\big(\tilde{\mu}_{m,n}(i, c_1), \dots, \tilde{\mu}_{m,n}(i, c_m)\big).$$

Finally, for a threshold $\alpha \in [0,1]$, idea i "advances" if

$$\widetilde{r}_{m,n}(i) \ \cap\ [\alpha, 1] \ \neq\ \varnothing.$$

Example 11. (Real-World $(3,2)$-SuperHyperFuzzy Front-End: Evaluating Green Technology Ideas)**.** Let

$$\mathcal{I} = \{\text{DroneDelivery, SolarRefrigerator}\}$$

be two proposed sustainability-focused project ideas, and let the evaluation criteria be

$$\mathcal{C} = \{\text{Technical, Environmental, Market}\},$$

so $m = 3$. We fix $n = 2$. Denote by $\mathcal{P}_2^*([0,1])$ the family of all nonempty families of nonempty subsets of $[0,1]$. We define the superhyper-membership

$$\tilde{\mu}_{3,2} : \mathcal{I} \times \mathcal{C} \ \longrightarrow\ \mathcal{P}_2^*([0,1])$$

by expert panels and stakeholder surveys as follows:

DroneDelivery:

$$\tilde{\mu}_{3,2}(\text{DroneDelivery, Technical}) = \big\{\{0.7, 0.8\},\ \{0.75, 0.85\}\big\},$$
$$\tilde{\mu}_{3,2}(\text{DroneDelivery, Environmental}) = \big\{\{0.6, 0.7\},\ \{0.65, 0.75\}\big\},$$
$$\tilde{\mu}_{3,2}(\text{DroneDelivery, Market}) = \big\{\{0.8, 0.9\},\ \{0.85, 0.95\}\big\}.$$

SolarRefrigerator:

$$\tilde{\mu}_{3,2}(\text{SolarRefrigerator, Technical}) = \big\{\{0.6, 0.7\},\ \{0.65, 0.75\}\big\},$$
$$\tilde{\mu}_{3,2}(\text{SolarRefrigerator, Environmental}) = \big\{\{0.9, 1.0\},\ \{0.85, 0.95\}\big\},$$
$$\tilde{\mu}_{3,2}(\text{SolarRefrigerator, Market}) = \big\{\{0.5, 0.6\},\ \{0.55, 0.65\}\big\}.$$

Choose the classical fuzzy aggregator

$$F(x,y,z) \;=\; 0.4\,x \;+\; 0.3\,y \;+\; 0.3\,z.$$

The induced superhyper-aggregation $\widetilde{F}_{3,2} : (\mathcal{P}_2^*([0,1]))^3 \to \mathcal{P}_2^*([0,1])$ is given by

$$\widetilde{F}_{3,2}(A,B,C) = \big\{\, F(x,y,z) \mid x \in A',\, y \in B',\, z \in C' \big\}_2^*,$$

where $\{\cdots\}_2^*$ denotes taking all nonempty pairs of results.

Then the readiness superhyper-sets are:

DroneDelivery:

$$\widetilde{r}_{3,2}(\text{DroneDelivery}) = \widetilde{F}_{3,2}\big(\{0.7,0.8\},\{0.6,0.7\},\{0.8,0.9\}\big) = \big\{\{0.76,0.82\},\,\{0.80,0.86\}\big\}.$$

SolarRefrigerator:

$$\widetilde{r}_{3,2}(\text{SolarRefrigerator}) = \widetilde{F}_{3,2}\big(\{0.6,0.7\},\{0.9,1.0\},\{0.5,0.6\}\big) = \big\{\{0.69,0.75\},\,\{0.73,0.79\}\big\}.$$

For a development threshold $\alpha = 0.80$, an idea advances if its readiness superhyper-set intersects $[\alpha, 1]$. We see:

$$\widetilde{r}_{3,2}(\text{DroneDelivery}) \cap [0.80,1] = \{0.80, 0.82, 0.86\} \neq \varnothing,$$
$$\widetilde{r}_{3,2}(\text{SolarRefrigerator}) \cap [0.80,1] = \varnothing.$$

Therefore, only *DroneDelivery* proceeds to formal development under this $(3,2)$-SuperHyperFuzzy Front-End evaluation.

Example 12 (Real-World $(2,2)$-SuperHyperFuzzy Front-End: Public Health Intervention Screening)**.** Consider two proposed COVID-19 mitigation strategies:

$$\mathcal{I} = \{\text{Mask Mandate},\ \text{Vaccine Incentive}\},$$

evaluated on two criteria:

$$\mathcal{C} = \{\text{Effectiveness},\ \text{Public Acceptance}\},$$

so $m = 2$, and set $n = 2$. We elicit two independent assessments for each criterion—an expert panel and a simulation model—each producing a range of membership degrees. Thus the superhyper-membership function

$$\tilde{\mu}_{2,2} : \mathcal{I} \times \mathcal{C} \;\longrightarrow\; \mathcal{P}_2^*([0,1])$$

is defined by:

Mask Mandate:

$$\tilde{\mu}_{2,2}(\text{Mask Mandate}, \text{Effectiveness}) = \big\{\{0.80,0.85\},\,\{0.75,0.90\}\big\},$$
$$\tilde{\mu}_{2,2}(\text{Mask Mandate}, \text{Public Acceptance}) = \big\{\{0.60,0.70\},\,\{0.55,0.65\}\big\}.$$

Vaccine Incentive:

$$\tilde{\mu}_{2,2}(\text{Vaccine Incentive}, \text{Effectiveness}) = \big\{\{0.90,0.95\},\,\{0.85,0.90\}\big\},$$
$$\tilde{\mu}_{2,2}(\text{Vaccine Incentive}, \text{Public Acceptance}) = \big\{\{0.70,0.80\},\,\{0.65,0.75\}\big\}.$$

Choose the fuzzy aggregator

$$F(x,y) \;=\; 0.7\,x \;+\; 0.3\,y.$$

Define the superhyper-aggregation

$$\widetilde{F}_{2,2}(A,B) = \big\{\, F(x,y) \mid x \in A',\ y \in B' \big\}_2^*,$$

where each $A' \in A \subseteq \mathcal{P}_1^*([0,1])$ and similarly for B, and $\{\cdots\}_2^*$ takes the nonempty second-powerset closure. Then the readiness superhyper-sets are:

$$\widetilde{r}_{2,2}(\text{Mask Mandate}) = \widetilde{F}_{2,2}\big(\{0.80,0.85\},\{0.60,0.70\}\big)$$

$$= \big\{\{0.74, 0.77, 0.745, 0.805\},\ \{0.7775, 0.72, 0.795, 0.825\}\big\},$$

$$\widetilde{r}_{2,2}(\text{Vaccine Incentive}) = \widetilde{F}_{2,2}\big(\{0.90, 0.95\},\ \{0.70, 0.80\}\big)$$
$$= \big\{\{0.84, 0.87, 0.875, 0.905\},\ \{0.79, 0.82, 0.825, 0.855\}\big\}.$$

For a cutoff $\alpha = 0.88$, an intervention "advances" if its readiness superhyper-set intersects $[\alpha, 1]$. We find:

$$\widetilde{r}_{2,2}(\text{Mask Mandate}) \cap [0.88, 1] = \varnothing, \quad \widetilde{r}_{2,2}(\text{Vaccine Incentive}) \cap [0.88, 1] = \{0.875, 0.905\} \neq \varnothing.$$

Hence only *Vaccine Incentive* proceeds to formal policy development under this $(2, 2)$-SuperHyperFuzzy Front-End evaluation.

Theorem 21. *Let $(\mathcal{I}, \mathcal{C}, \tilde{\mu}_{m,n}, F)$ be an (m, n)-SuperHyperFuzzy Front-End. Then:*

(1) ***Generalization of Fuzzy Front-End.*** *If $n = 0$ and each $\tilde{\mu}_{m,0}(i, c_j)$ is a singleton $\{\mu_0(i, c_j)\} \subseteq [0, 1]$, then $\tilde{\mu}_{m,0}$ reduces to an ordinary membership function μ_0, and $\widetilde{r}_{m,0}(i) = \{r_0(i)\}$ with $r_0(i) = F(\mu_0(i, c_1), \ldots, \mu_0(i, c_m))$.*

(2) ***Generalization of HyperFuzzy Front-End.*** *If $m = 1$ and $n = 1$, then $\tilde{\mu}_{1,1} : \mathcal{I} \times \mathcal{C} \to \tilde{\mathcal{P}}([0, 1])$ is exactly a HyperFuzzy Front-End membership, and $\widetilde{r}_{1,1}(i)$ coincides with the HyperFuzzy readiness hyper-set.*

(3) (m, n)***-SuperHyperFuzzy Set Structure.*** *Identify the universe $X = \mathcal{I} \times \mathcal{C}$. Then*

$$\tilde{\mu}_{m,n} : X \longrightarrow \mathcal{P}_n^*([0, 1])$$

is precisely an (m, n)-SuperHyperFuzzy Set on X with codomain $[0, 1]$ in the sense of Definition 1.7.

Proof: (1) If $n = 0$, then $\mathcal{P}_0^*([0, 1]) = [0, 1]$. Hence $\tilde{\mu}_{m,0}$ and $\widetilde{r}_{m,0}$ collapse to single-valued functions, reproducing the classical Fuzzy Front-End.

(2) For $m = n = 1$, we have $\mathcal{P}_1^*([0, 1]) = \tilde{\mathcal{P}}([0, 1])$. Thus $\tilde{\mu}_{1,1}$ assigns each (i, c_j) a nonempty subset of $[0, 1]$, exactly the HyperFuzzy Front-End structure, and $\widetilde{r}_{1,1}$ matches the HyperFuzzy aggregation.

(3) By letting $X = \mathcal{I} \times \mathcal{C}$, the map $\tilde{\mu}_{m,n} : X \to \mathcal{P}_n^*([0, 1])$ is in one-to-one correspondence with an (m, n)-SuperHyperFuzzy Set on X, per Definition 1.7. No further axioms are needed.

□

Theorem 23. (Reduction to Classical Fuzzy Front-End)**.** *Let $\big(\mathcal{I}, \mathcal{C}, \tilde{\mu}_{m,n}, F\big)$ be an (m, n)-SuperHyperFuzzy Front-End. Suppose $n = 0$ and for each (i, c_j) the set $\tilde{\mu}_{m,0}(i, c_j) \subseteq [0, 1]$ is a singleton $\{\mu_0(i, c_j)\}$. Then*

$$\tilde{\mu}_{m,0} : \mathcal{I} \times \mathcal{C} \longrightarrow [0, 1], \quad \tilde{\mu}_{m,0}(i, c_j) = \mu_0(i, c_j),$$

and the induced readiness superhyper-set $\widetilde{r}_{m,0}(i) \subseteq [0, 1]$ satisfies

$$\widetilde{r}_{m,0}(i) = \big\{ F\big(\mu_0(i, c_1), \ldots, \mu_0(i, c_m)\big)\big\},$$

recovering exactly the classical Fuzzy Front-End readiness score.

Proof: When $n = 0$, $\mathcal{P}_0^*([0, 1]) = [0, 1]$, so $\tilde{\mu}_{m,0}$ is single-valued. By definition

$$\widetilde{F}_{m,0}\big(\{\mu_0(i, c_1)\}, \ldots, \{\mu_0(i, c_m)\}\big) = \big\{ F(x_1, \ldots, x_m) \mid x_j = \mu_0(i, c_j)\big\}_0^* = \{ F(\mu_0(i, c_1), \ldots, \mu_0(i, c_m))\},$$

since the nonempty 0th-powerset closure of a singleton is itself. But this is exactly the crisp readiness score of the classical Fuzzy Front-End. □

Theorem 24. (Reduction to HyperFuzzy Front-End)**.** *Let $\big(\mathcal{I}, \mathcal{C}, \tilde{\mu}_{m,n}, F\big)$ be an (m, n)-SuperHyperFuzzy Front-End. If $n = 1$, then $\mathcal{P}_1^*([0, 1]) = \{A \subseteq [0, 1] \mid A \neq \emptyset\}$. Hence*

$$\tilde{\mu}_{m,1} : \mathcal{I} \times \mathcal{C} \longrightarrow \tilde{\mathcal{P}}([0, 1]),$$

and the induced $\widetilde{r}_{m,1}(i) \subseteq [0, 1]$ coincides with the HyperFuzzy Front-End readiness hyper-set:

$$\widetilde{r}_{m,1}(i) = \{ F(x_1, \ldots, x_m) \mid x_j \in \tilde{\mu}_{m,1}(i, c_j)\}.$$

Proof: Since $n = 1$, by definition $\mathcal{P}_1^*([0,1]) = \tilde{\mathcal{P}}([0,1])$. Thus $\tilde{\mu}_{m,1}$ assigns each (i, c_j) a nonempty subset of $[0,1]$. The superhyper-aggregation reduces to

$$\widetilde{F}_{m,1}(A_1, \ldots, A_m) = \{\, F(x_1, \ldots, x_m) \mid x_j \in A_j \},$$

which is precisely the induced hyper-readiness set in a HyperFuzzy Front-End. □

Theorem 25. (Monotonicity)**.** *Assume the aggregator $F : [0,1]^m \to [0,1]$ is non-decreasing in each coordinate. Let*

$$(A_1, \ldots, A_m),\ (B_1, \ldots, B_m)\ \in\ \big(\mathcal{P}_n^*([0,1])\big)^m$$

satisfy $A_j \subseteq B_j$ for all $1 \le j \le m$. Then

$$\widetilde{F}_{m,n}(A_1, \ldots, A_m)\ \subseteq\ \widetilde{F}_{m,n}(B_1, \ldots, B_m).$$

In particular, for any $i \in \mathcal{I}$, if $\tilde{\mu}_{m,n}(i, c_j) \subseteq \tilde{\nu}_{m,n}(i, c_j)$ for all j, then $\widetilde{r}^{\mu}_{m,n}(i) \subseteq \widetilde{r}^{\nu}_{m,n}(i)$.

Proof: By definition,

$$\widetilde{F}_{m,n}(A_1, \ldots, A_m) = \big\{\, F(x_1, \ldots, x_m) \mid x_j \in A_j \big\}_n^*,$$

and similarly for $(B_1, \ldots, B_m)$. Since each $A_j \subseteq B_j$, every tuple $(x_1, \ldots, x_m)$ with $x_j \in A_j$ also satisfies $x_j \in B_j$. Hence

$$\{\, F(x_1, \ldots, x_m) \mid x_j \in A_j \}\ \subseteq\ \{\, F(x_1, \ldots, x_m) \mid x_j \in B_j \}.$$

Taking the nonempty nth-powerset closure preserves inclusion:

$$\big\{\cdots\big\}_n^*\ \subseteq\ \big\{\cdots\big\}_n^*.$$

Thus $\widetilde{F}_{m,n}(A_1, \ldots, A_m) \subseteq \widetilde{F}_{m,n}(B_1, \ldots, B_m)$. The statement about readiness sets follows by setting $A_j = \tilde{\mu}_{m,n}(i, c_j)$ and $B_j = \tilde{\nu}_{m,n}(i, c_j)$. □

Theorem 26. (Idempotence)**.** *Suppose F satisfies $F(x, \ldots, x) = x$ for all $x \in [0,1]$. Then for any nonempty nth-powerset family $A \subseteq \mathcal{P}_{n-1}^*([0,1])$,*

$$\widetilde{F}_{m,n}(A, A, \ldots, A) = A_n^*,$$

where A_n^* denotes the nonempty nth-powerset closure of A. In particular, if $\tilde{\mu}_{m,n}(i, c_j) = A$ for all j, then $\widetilde{r}_{m,n}(i) = A_n^*$.

Proof: By definition,

$$\widetilde{F}_{m,n}(A, \ldots, A) = \big\{\, F(x_1, \ldots, x_m) \mid x_1, \ldots, x_m \in A \big\}_n^*.$$

But $F(x_1, \ldots, x_m) = x_1$ whenever all coordinates equal x_1 and F is idempotent; more generally, since all $x_j \in A$, the set $\{\, F(x_1, \ldots, x_m) \mid x_j \in A\}$ coincides with A itself. Therefore

$$\widetilde{F}_{m,n}(A, \ldots, A) = A_n^*,$$

as claimed. The statement for $\widetilde{r}_{m,n}(i)$ is immediate by taking each hyper-membership equal to A. □

4 | Conclusion

In this paper, we have extended the concepts of Fuzzy Trace and Fuzzy Front–End using HyperFuzzy and SuperHyperFuzzy sets, and provided a brief examination of their properties with illustrative examples. In future work, we plan to investigate algorithmic implementations and conduct further evaluations of these concepts' effectiveness. We also intend to extend our definitions by incorporating Neutrosophic Sets [31, 32], Refined Neutrosophic Sets [33, 34], Uncertain Sets [35, 36], Plithogenic Sets [37, 38, 39], and related frameworks.

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Data Availability

This manuscript presents purely conceptual work without empirical data. Scholars interested in these ideas are invited to undertake experimental or case-study research to substantiate and extend the proposed frameworks.

Ethical Approval

This paper involves no human or animal subjects and thus did not require ethics committee review or approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Conflicts of Interest

The authors declare that there are no competing interests concerning the content or publication of this article.

Disclaimer

The theoretical models and propositions herein have not yet been subjected to practical validation. Readers should independently verify all citations and be aware that inadvertent inaccuracies may remain. The opinions expressed are those of the authors and do not necessarily represent the views of affiliated organizations.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. *Information and control,* 8(3), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] Akram, M. (2011). Bipolar fuzzy graphs. *Information sciences,* 181(24), 5548-5564. https://doi.org/10.1016/j.ins.2011.07.037

[3] Atanassov, K. T. (2020). Circular intuitionistic fuzzy sets. *Journal of intelligent & fuzzy systems,* 39(5), 5981-5986. https://doi.org/10.3233/JIFS-189072

[4] Broumi, S., Talea, M., Bakali, A., & Smarandache, F. (2016). Single valued Neutrosophic graphs. *Journal of new theor*y, (10), 86 -101. https://izlik.org/JA44RD85ED

[5] Xu, Z. (2014). *Hesitant fuzzy sets theory.* Cham: Springer International Publishing. https://doi.org/10.1007/978-3-319-04711-9

[6] Radha, R., Mary, A. S. A., & Smarandache, F. (2021). Quadripartitioned Neutrosophic pythagorean soft set. *International journal of Neutrosophic science (IJNS), 14*(1), 9- 23. https://doi.org/10.5281/zenodo.4678763

[7] Chatterjee, T., & Pramanik, S. (2024). Triangular fuzzy quadripartitioned Neutrosophic set and its properties. *Neutrosophic sets and systems,* 75(1), 2. https://digitalrepository.unm.edu/nss_journal/vol75/iss1/2

[8] Akram, M., Saleem, D., & Al-Hawary, T. (2020). Spherical fuzzy graphs with application to decision-making. *Mathematical and computational applications,* 25(1), 8. https://doi.org/10.3390/mca25010008

[9] Smarandache, F. (2018). Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and Neutrosophic sets-revisited. *Neu-trosophic sets and systems, 21*(1), 153–166. https://fs.unm.edu/nss8/index.php/111/article/view/316

[10] Sultana, F., Gulistan, M., Ali, M., Yaqoob, N., Khan, M., Rashid, T., & Ahmed, T. (2022). A study of plithogenic graphs: Ap-plications in spreading coronavirus disease (Covid-19) globally. *Journal of ambient intelligence and humanized computing, 14* (10), 13139 -13159. https://doi.org/10.1007/s12652-022-03772-6

[11] Jun, Y. B., Hur, K., & Lee, K. J. (2017). Hyperfuzzy subalgebras of Bck/Bci-algebras. *Annals of fuzzy mathematics and infor-matics, 15* (1),17-28. https://doi.org/10.30948/afmi.2018.15.1.17
[12] Song, S. Z., Kim, S. J., & Jun, Y. B. (2017). Hyperfuzzy ideals in Bck/Bci-algebras. *Mathematics, 5*(4), 81. https://doi.org/10.3390/math5040081
[13] Fujita, T., & Smarandache, F. (2025). Hierarchical uncertainty modeling via (m, n)-superhyperuncertainand (h, k)-ary (m, n)- superhyperuncertain sets: Unified extensions of fuzzy, Neutrosophic, soft, rough, and plithogenic set theories. *European journal of pure and applied mathematics, 18*(4), 6834-6834. https://doi.org/10.29020/nybg.ejpam.v18i4.6834
[14] Fujita, T. (2025). *Advancing uncertain combinatorics through graphization, hyperization, and uncertainization: Fuzzy, Neutrosophic, soft, rough, and beyond.* Biblio Publishing. file:///C:/Users/Admin/Downloads/Fujita-Neutro-GraphV2.pdf
[15] Hjorth, G. (2005). T. Jech. Set theory. The third millennium edition, revised and expanded. Springer-Verlag, Berlin, 2003, viii+769 pp. *Bulletin of symbolic logic, 11*(2), 243-245. https://doi.org/10.1017/S1079898600003358
[16] Smarandache, F. (2024). Foundation of superhyperstructure & Neutrosophic superhyperstructure. *Neutrosophic sets and systems, 63*(2024), 367-381. https://doi.org/10.5281/zenodo.10535374
[17] Nazari, Z., & Mosapour, B. (2018). The entropy of hyperfuzzy sets. *Journal of dynamical systems and geometric theories, 16* (2), 173-185. https://doi.org/10.1080/1726037X.2018.1436270
[18] Smarandache, F. (2017). HyperUncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review, 14*, 10–19. https://fs.unm.edu/CR/HyperUncertain-SuperUncertain-SuperHyperUncertain.pdf
[19] Reyna, V. F., & Brainerd, C. J. (1998). Fuzzy-trace theory and false memory: New frontiers. *Journal of experimental childpsychology, 71*(2), 194-209. https://doi.org/10.1006/jecp.1998.2472
[20] Blalock, S. J., & Reyna, V. F. (2016). Using fuzzy-trace theory to understand and improve health judgments, decisions, and behaviors: A literature review. *Health psychology, 35*(8), 781. https://psycnet.apa.org/doi/10.1037/hea0000384
[21] Reyna, V. F., & Brainerd, C. J. (1995). Fuzzy-trace theory: An interim synthesis. *Learning and individual differences, 7*(1), 1-75. https://doi.org/10.1016/1041-6080(95)90031-4
[22] Brainerd, C. J., & Reyna, V. F. (1998). Fuzzy-trace theory and children's false memories. *Journal of experimental child psychology, 71*(2), 81-129. https://doi.org/10.1006/jecp.1998.2464
[23] Brainerd, C. J., Bialer, D. M., & Chang, M. (2022). Fuzzy-trace theory and false memory: Meta-analysis of conjoint recognition. *Journal of experimental psychology: Learning, memory, and cognition, 48*(11), 1680. https://psycnet.apa.org/doi/10.1037/xlm0001040
[24] Morrow, D., Azevedo, R. F. L., Garcia-Retamero, R., Hasegawa-Johnson, M., Huang, T., Schuh, W., ... & Zhang, Y. (2019). Contextualizing numeric clinical test results for gist comprehension: Implications for EHR patient portals. *Journal of experimental psychology: Applied, 25*(1), 41. https://psycnet.apa.org/doi/10.1037/xap0000203
[25] Koen, P. A., Ajamian, G. M., Boyce, S., Clamen, A., Fisher, E., Fountoulakis, S., Puri, P., & Seibert, R. (2002). Fuzzy front end: Effective methods, tools, and techniques. In *The PDMA toolbook for new product development* (pp. 5–35). John Wiley & Sons. https://www.wi-ley.com/en-us/The+PDMA+ToolBook+for+New+Product+Development-p-9780471236628
[26] Khurana, A., & Rosenthal, S. R. (1997). Integrating the fuzzy front end of new product development. *IEEE engineering management review, 25*(4), 35-49. https://sloanreview.mit.edu/article/integrating-the-fuzzy-front-end-of-new-product-development/
[27] Kim, J., & Wilemon, D. (2002). Focusing the fuzzy front–end in new product development. *R&d management, 32*(4), 269-279. https://doi.org/10.1111/1467-9310.00259
[28] Gassmann, O., & Schweitzer, F. (2014). *Management of the fuzzy front end of innovation*. Springer Cham. https://doi.org/10.1007/978-3-319-01056-4
[29] Herstatt, C., & Verworn, B. (2004). The 'fuzzy front end'of innovation. In *Bringing technology and innovation into the boardroom: Strategy, innovation and competences for business value* (pp. 347-372). London: Palgrave Macmillan UK. https://doi.org/10.1057/9780230512771_16
[30] Gassmann, O., & Schweitzer, F. (2013). Managing the unmanageable: The fuzzy front end of innovation. In *Management of the fuzzy front end of innovation* (pp. 3-14). Cham: Springer International Publishing. https://doi.org/10.1007/978-3-319-01056-4_1
[31] Berrezueta, R. A., Aldaz, E. M., López, B. C. E., & Aguirre, J. A. A. (2023). Neutrosophic assessment of personality traits. *Neu-trosophic sets and systems, 62*(1), 17. https://digitalrepository.unm.edu/nss_journal/vol62/iss1/17/
[32] Zhang, H. (2025). Neutrosophic approaches for analyzing the interaction between the sports industry and community health: A multi-attribute decision-making perspective. *Neutrosophic sets and systems, 77*(1), 20–40. https://digitalrepository.unm.edu/nss_journal/vol77/iss1/2
[33] Alhasan, Y. A., & Abdulfatah, R. A. (2023). Division of refined Neutrosophic numbers. *Neutrosophic sets and systems, 60*, 1–5. https://doi.org/10.5281/zenodo.10224078
[34] Smarandache, F. (2013). n-Valued refined Neutrosophic logic and its applications to physics. *Progress in physics, 4*, 143-146. https://doi.org/10.5281/zenodo.30245
[35] Fujita, T., & Smarandache, F. (2025). A unified framework for U-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models. *Neutrosophic sets and systems, 91*, 337–380. https://doi.org/10.5281/zenodo.16754933
[36] Fujita, T., & Smarandache, F. (2025). *A dynamic survey of fuzzy, intuitionistic fuzzy, Neutrosophic, plithogenic, and extensional*.SetsNeutrosophic Science International Association. https://pdfhost.io/v/HaB6hdkPHH_A_Dynamic_Survey_of_Fuzzy__Intuitionistic_Fuzzy__Neutrosophic__Plithogenic__and_Extensional_Sets
[37] Ramos, E. L. H., Ayala, L. R. A., & Macas, K. A. S. (2025). Study of factors that influence a victim's refusal to testify for sexual reasons due to external influence using plithogenic n-superhypergraphs. *Operational research journal, 46*(2), 328–337. https://rev-inv-ope.pantheonsor-bonne.fr/sites/default/files/inline-files/4602-19STUDY%20OF%20FACTORS%20THAT%20INFLUENCE%20A %20VICTIM%27S %20RE-FUSAL%20TO %20TESTIFY%20FOR%20SEXUAL%20REASONS%20DUE%20TO%20EX-TERNAL %20INFLUENCE%20USING %20PLYTHOGENIC%20N-SUPERHYPERGRAPHS_0.pdf
[38] SmarandacheÂ¹, F. (2018). Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems: An international book series in information science and engineering, 22/2018*(22), 168-170. https://fs.unm.edu/NSS/ExtensionOfSoftSetToHypersoftSet.pdf
[39] Smarandache, F. (2020). Extension of hypergraph to n-superhypergraph and to Plithogenic n-superhypergraph, and extension of hyperalgebra to n-ary (classical-/neutro-/anti-) hyperalgebra. *Neutrosophic sets and systems, 33*, 290–296. https://doi.org/10.5281/zenodo.3783103